CALCULUS II
INVERSE HYPERBOLIC FUNCTIONS

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It is possible to write the inverse hyperbolic functions as natural logarithms of algebraic expressions. To do this, we write e^x in terms of the hyperbolic function, then take the logarithm of both sides.

Let $y = \operatorname{arcsinh} x$. Then $x = \sinh y$. Now

$$e^y = \sinh y + \cosh y = \sinh y + \sqrt{\sinh^2 y + 1} = x + \sqrt{x^2 + 1},$$

so

$$\boxed{\operatorname{arcsinh} x = \log(x + \sqrt{x^2 + 1}).}$$

Let $y = \operatorname{arccosh} x$. Then $x = \cosh y$. Now

$$e^y = \cosh y + \sinh y = \cosh y + \sqrt{\cosh^2 y - 1} = x + \sqrt{x^2 - 1},$$

so

$$\boxed{\operatorname{arccosh} x = \log(x + \sqrt{x^2 - 1}).}$$

Let $y = \operatorname{arctanh} x$, so that $x = \tanh y$. Let $u = e^y$, so that $\log u = y = \operatorname{arctanh} x$. Then

$$x = \frac{u - u^{-1}}{u + u^{-1}} = \frac{u^2 - 1}{u^2 + 1}.$$

We wish to solve this equation to make x a function of u^2. Multiplying by $u^2 + 1$, we obtain $xu^2 + x = u^2 - 1$. Then $(x-1)u^2 = -x - 1$, so $(1-x)u^2 = 1 + x$, whence $u^2 = \frac{1+x}{1-x}$. Taking the natural logarithm of both sides gives

$$\boxed{\operatorname{arctanh} x = \frac{1}{2} \log\left(\frac{1+x}{1-x}\right).}$$

Let $y = \operatorname{arccoth} x$, so that $x = \coth y$. Let $u = e^y$, so that $\log u = y = \operatorname{arccoth} x$. Then

$$x = \frac{u + u^{-1}}{u - u^{-1}} = \frac{u^2 + 1}{u^2 - 1}.$$

Then $xu^2 - x = u^2 + 1$, so $(x-1)u^2 = x + 1$; thus $u^2 = \frac{x+1}{x-1}$. Taking the logarithm of both sides gives

$$\boxed{\operatorname{arccoth} x = \frac{1}{2} \log\left(\frac{x+1}{x-1}\right)}.$$

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Let $y = \operatorname{arcsech} x$, so that $x = \operatorname{sech} y$. Let $u = e^y$, so that $\log u = y = \operatorname{arcsech} x$. Then

$$x = \frac{2}{u + u^{-1}} = \frac{2u}{u^2 + 1}.$$

Then $xu^2 - 2u + x = 0$, so by the quadratic formula,

$$u = \frac{2 \pm \sqrt{4 - 4x^2}}{2x} = \frac{1 \pm \sqrt{1 - x^2}}{x}.$$

Since u is always positive, the negative radical solution is spurious. Taking the logarithm of both sides gives

$$\boxed{\operatorname{arcsech} x = \log\Big(\frac{1 + \sqrt{1 - x^2}}{x}\Big).}$$

Let $y = \operatorname{arccsch} x$, so that $x = \operatorname{csch} y$. Let $u = e^y$, so that $\log u = y = \operatorname{arccsch} x$. Then

$$x = \frac{2}{u - u^{-1}} = \frac{2u}{u^2 - 1}.$$

Then $xu^2 - 2u - x = 0$, so by the quadratic formula,

$$u = \frac{2 \pm \sqrt{4 + 4x^2}}{2x} = \frac{1 \pm \sqrt{1 + x^2}}{x}.$$

Since u is always positive, the negative radical solution is spurious. Taking the logarithm of both sides gives

$$\boxed{\operatorname{arccsch} x = \log\Big(\frac{1 + \sqrt{1 + x^2}}{x}\Big).}$$

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